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A System of Nonlinear Partial Differential Equations Describing Cylindrical Plasma Collapse

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A SYSTEM OF NONLINEAR PARTIAL

DIFFERENTIAL EQUATIONS DESCRIBING CYLINDRICAL PLASMA COLLAPSE

by

Alfred Carasso and B. R. Suydam

ABSTRACT

We consider the snow-plough model for describing cylindrical plasma collapse for a specified constant driving term. This coupled nonlinear system consists of five partial differential equations in two independent variables, one of which is the time variable. Generally, the initial value problem for similar systems is improperly posed. However, here we show that by direct construction of the unique solution, explicitly in terms of the initial data, the solution exists for all positive times and is generally an infinitely differentiable function of the independent variables. Nevertheless, the solution always develops a nonphysical singularity after a certain positive time, and thereafter ceases to describe the underlying physical situation. Our theory leads to an *a priori* bound in terms of the initial data, on the time interval during which the snow-plough model is physically realistic. We discuss several examples which illustrate the pathologies exhibited by the solution.

1. INTRODUCTION

The underlying physical problem considered in this report is the compression of a fully ionized plasma by a magnetic piston. Because a fully ionized plasma is normally a very good conductor, we may set the resistivity equal to zero. The magnetic field then cannot penetrate into the plasma; it simply drives the sharp plasma-vacuum interface like a piston. To treat this problem properly, one should solve the hydromagnetic equations in the plasma for the magnetically driven shock. This shock ultimately reaches the center of the plasma and is reflected back to the piston. In this report we are concerned with the early phase of the compression, that is, before arrival of the back shock. In this case, the piston motion can be well approximated by a simplified model, the "snow-plough" equations,[1,2] and one can avoid the more formidable problem posed by the full hydrodynamical equations. Thus, instead of following the development of the shock, we assume that each plasma element remains undisturbed until the piston arrives. When the piston arrives, each plasma element is picked up and sticks to the piston face. We thus imagine the shock to remain infinitesimally close to the piston in this approximation and assume the shock compression of the plasma is infinite.

Although these approximations are somewhat crude, the snow-plough model has proved useful in plasma compression calculations. Recently, Nelson, Brown, and Hart[2] described a code used for numerical computations of such problems. They reported that the code, based partly on the snow-plough model, runs well and gives good results.

In Sec. 2, we derive the snow-plough equations directly from the principles of mass and momentum conservation for the two-dimensional case of a cylindrical plasma. These equations form a nonlinear system of five partial differential equations in two independent variables. This set of equations can also be obtained from the full hydrodynamical equations as a limiting case, when the temperature approaches zero and γ (the ratio of specific heats) approaches unity.

This simplified system of snow-plough equations is the basis of our study. These equations are highly unconventional, and the model raises new mathematical questions. The initial value problem for similar systems is generally <u>improperly</u> <u>posed</u>. In fact, a linearized stability analysis of the snow-plough equations reveals that the growth rate of perturbations becomes infinite as their wavelength approaches zero. Yet, surprisingly, the algorithm discussed in Ref. 2 encountered no stability difficulties—an apparent contradiction to a well-known principle in numerical analysis.[3] In this report we take the first step toward answering these questions. We shall show that for a specified constant driving term, the <u>nonlinear</u> initial value problem is <u>well-posed</u>, and we shall construct its unique solution explicitly in terms of the initial data. The solution will exist for all $t \geq 0$. Nevertheless, as will be shown, the solution <u>always</u> develops a nonphysical singularity after a certain positive time, T_c, and thereafter ceases to describe the underlying physical situation. This is so despite the fact that the solution is generally an infinitely differentiable function of the independent variables. In fact, our construction leads to an upper bound, which may be calculated explicitly in terms of the initial data, for the time interval during which the snow-plough model is physically realistic. In Sec. 5, we discuss examples which illustrate these points, as well as further complications not covered by our theorems.

The above problem is a particular instance of the more general problem of moving a simple, closed plane curve according to some prescription. This type of problem occurs in various physical situations. For example, optical problems may be formulated in this fashion, the plane curve (or, more generally, surface) being an isophase front. With proper rules for moving the curve, diffraction and even nonlinear optical effects can be fully accounted for; also, such a formulation has attractive computational features. Our ultimate aims, therefore, are broader than the snow-plough problem discussed in this report.

As a simple example of the above general class of problems, consider the evolution of a curve when each point on the curve is moved toward the instantaneous inward normal, at a uniform velocity, which we may normalize to unity. Let the curve be given parametrically by

$$x = X(\lambda,t),\ y = Y(\lambda,t), \tag{1.1}$$

and let the element of arc length, ds, be given by

$$ds = Sd\lambda,\ S \equiv \left[\left(\partial_\lambda X\right)^2 + \left(\partial_\lambda Y\right)^2\right]^{1/2}, \tag{1.2}$$

where $\partial_\lambda \equiv \dfrac{\partial}{\partial\lambda}$.

Then, the equations of motion for the curve are

$$\partial_t X = -\left(1/S\right) \partial_\lambda Y \ , \quad \partial_t Y = \left(1/S\right) \partial_\lambda X \ . \tag{1.3}$$

The initial value problem is well-posed for this nonlinear system. However, if S were simply a function of (X,Y,λ,t), the initial value problem would probably be ill-posed because these equations constitute a generalization of the Cauchy-Riemann equations.

Although we have not found a method of generally classifying such systems, these problems have the following property in common. In their natural form, the equations are not quasi-linear. When, by introducing new dependent variables, they are quasi-linearized as

$$A\partial_t \vec{X} + B\partial_\lambda \vec{X} = C \ , \tag{1.4}$$

the matrices A, B, and C are highly singular.

2. DERIVATION OF THE SNOW-PLOUGH EQUATIONS

Let (x^1,x^2,x^3) denote Cartesian coordinates. We consider a cylindrical plasma with its axis along the x^3 coordinate so that

$$\frac{\partial}{\partial x^3} \text{(anything)} = 0 \ . \tag{2.1}$$

The plasma-vacuum interface can be described as a simple closed curve, $\Gamma = \Gamma(t)$, in the (x^1,x^2) plane. This curve is given parametrically by

$$x^1 = R(x,t) \tag{2.2}$$

and

$$x^2 = Z(x,t), \tag{2.3}$$

where t is time and x is a conveniently chosen parameter such that $0 \le x \le 2\pi$. Thus, R and Z are 2π-periodic functions of x. We choose x to increase anti-clockwise around Γ. The unit tangent $\vec{\tau}(x,t)$ and the unit normal $\vec{n}(x,t)$ to Γ are given by

$$\tau_1 = \left(1/S\right) R_x, \quad \tau_2 = \left(1/S\right) Z_x, \tag{2.4}$$

and

$$n_1 = -\left(1/S\right) Z_x, \quad n_2 = \left(1/S\right) R_x, \tag{2.5}$$

where

$$S = \left(R_x^2 + Z_x^2\right)^{1/2} . \tag{2.6}$$

The convention chosen for x makes $\vec{n}(x,t)$ point inward into the plasma. Let us revert momentarily to three dimensions and consider an element $\delta\Sigma$ of the interface with velocity $\vec{U}$. Then, in time δt, $\delta\Sigma$ sweeps up a volume $(\vec{U}\cdot\vec{n})\delta\Sigma$. Let ρ be the constant plasma volume density and μ be the mass per unit area of the piston, that is, mass of plasma already swept up. Then, because $\delta\Sigma = S\delta x^3 \delta x$, mass conservation is given by

$$M_t = \rho S(\vec{U}\cdot\vec{n}) \quad , \quad (M = \mu S). \tag{2.7}$$

Similarly, if $\Pi > 0$ is the jump in total pressure across the plasma-vacuum interface, momentum conservation is given by

$$(M\vec{U})_t = \vec{n}\Pi s. \tag{2.8}$$

With $\vec{U} = (U,V)$, we also have

$$R_t = U,\ Z_t = V\ . \tag{2.9}$$

Once Π is specified, Eqs. (2.7), (2.8), and (2.9) are the equations of motion of the piston, that is, the curve Γ . Here we consider only the case where Π is a constant specified a priori. Generally, the plasma may have a "frozen in" magnetic field, $B_{p\ell}$, in addition to its pressure p. Then, Π is given by

$$\Pi = \frac{1}{8\pi} \left[\left(B_{vac}\right)^2 - \left(B_{p\ell}\right)^2\right] - p\ , \tag{2.10}$$

where B_{vac} must be computed from given external currents and boundary conditions on the moving curve Γ . We hope to consider this latter problem in future work.

We choose appropriate units so that $\rho = \Pi = 1$. Our nonlinear system is then

$$\begin{cases} R_t = U \\ Z_t = V \\ (MU)_t = -Z_x \qquad , \quad 0 \le x \le 2\pi \quad , \ t > 0 \\ (MV)_t = R_x \\ M_t = VR_x - UZ_x \end{cases} \tag{2.11}$$

where all five unknowns are functions of x and t, and 2π periodic in the space variable x.

Here we describe the initial conditions to be adjoined to this system. Let ω denote a generic positive constant, not necessarily having the same value at different occurrences. We assume the initial interface to be a smooth, simple closed curve with a continuously turning tangent so that

$$\left[R_x(x,0)\right]^2 + \left[Z_x(x,0)\right]^2 \geq \omega > 0 , \qquad (2.12)$$

$0 \leq x \leq 2\pi$. We also assume the initial velocity to be a continuous function of position along this curve and to be directed inward into the plasma at every point (Fig. 1). Thus, with n_1 and n_2 as in (2.5),

$$n_1(x,0)U(x,0) + n_2(x,0)V(x,0) \geq \omega > 0, \qquad (2.13)$$

$0 \leq x \leq 2\pi$. Finally, because the mass swept up by the piston is initially zero,

$$M(x,0) = 0 , \quad 0 \leq x \leq 2\pi . \qquad (2.14)$$

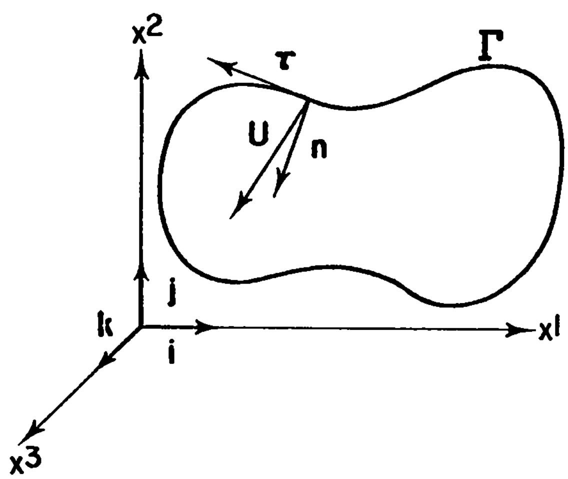

Fig. 1. Initial plasma-vacuum interface.

3. THE LINEARIZED INITIAL VALUE PROBLEM

It is instructive to study the evolutionarity of a linearized version of (2.11). Consider the last three equations in (2.11), namely,

$$\begin{cases} U_t = -\left(\frac{1}{M}\right) Z_x - \left(\frac{U}{M}\right) M_t \\ V_t = \left(\frac{1}{M}\right) R_x - \left(\frac{V}{M}\right) M_t \\ M_t = VR_x - UZ_x . \end{cases} \qquad (3.1)$$

By substituting the last equation for M_t in the first two equations, we obtain

$$\begin{cases} U_t = aZ_x - bR_x \\ V_t = bZ_x + cR_x \\ M_t = dZ_x + fR_x , \end{cases} \tag{3.2}$$

where $a = \left(\frac{U^2-1}{M}\right)$, $b = \frac{UV}{M}$, $c = \left(\frac{1-V^2}{M}\right)$, $d = V$, and $f = -U$.

A natural way of linearizing the system (2.11) is to consider the equations in (3.2) as being linear with the variable coefficients $a(x,t)$, $b(x,t)$, $c(x,t)$, $d(x,t)$, and $f(x,t)$ presumed known. In matrix-vector notation, with $\partial_x \equiv \frac{\partial}{\partial x}$, we may write the linearized problem as follows.

$$\begin{bmatrix} R \\ Z \\ U \\ V \\ M \end{bmatrix}_t = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -b\partial_x & a\partial_x & 0 & 0 & 0 \\ c\partial_x & b\partial_x & 0 & 0 & 0 \\ f\partial_x & d\partial_x & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ Z \\ U \\ V \\ M \end{bmatrix} . \tag{3.3}$$

Consider now the simplest case, where a, b, c, d, and f are constants. By Fourier transformation of the space variable x, we transform the above system into an equivalent system in Fourier space. At each point ξ, the transformed matrix, $A(\xi)$, is obtained from that in (3.3) by replacing ∂_x with $i\xi$. It is easy to find the eigenvalues of $A(\xi)$; they satisfy the characteristic equation

$$\lambda \left[\lambda^4 + \left(b^2 + ac\right) \xi^2\right] = 0. \tag{3.4}$$

Equation (3.4) shows that the linearized problem is well-posed if, and only if, $b^2 + ac = 0$; that is, if and only if,

$$U^2(x,t) + V^2(x,t) = 1 \ . \tag{3.5}$$

If $b^2 + ac \neq 0$, $A(\xi)$ always has one eigenvalue $\lambda(\xi)$, such that

$$\operatorname{Re} \lambda(\xi) = \sigma\sqrt{|\xi|} \ , \quad \sigma > 0. \tag{3.6}$$

In that case, a solution of (3.3) at $t > 0$ cannot be estimated in terms of the initial data in the L^2 norm, nor in most other useful metrics, and the initial value problem for (3.3) is <u>improperly</u> <u>posed</u>, much like the Cauchy problem for the Cauchy-Riemann equations.

In the next section we shall show that the nonlinear initial value problem (2.11) has a unique solution existing for all $t \geq 0$, and we shall construct this solution explicitly in terms of the initial data. The explicit form of the solution shows that the latter depends continuously on the data in the L^∞ norm.

4. THE NONLINEAR PROBLEM

For given $S > 0$, let J_S be the rectangle $[(x,t)|\ \ 0 \leq x \leq 2\pi,\ 0 < t < S]$ in the (x,t) plane. Let $\alpha(J_S)$ be the linear space of all real valued functions, $u(x,t)$, defined and of class C^1 on $\overline{J}_S$, the closure of J_S, and which are 2π periodic in the space variable x.

The nonlinear problem may be formulated analytically as follows. Given the initial values $U(x,0)$, $V(x,0)$, $R(x,0)$, $Z(x,0)$, and $M(x,0)$, which satisfy (2.12), (2.13), and (2.14), find a time interval $0 \leq t < T$ and five functions $U(x,t)$, $V(x,t)$, $R(x,t)$, $Z(x,t)$, and $M(x,t)$ such that

(a) $U, V, R, Z, M \ \varepsilon\ \alpha(J_T)$

(b) U, V, R, Z, M satisfy (2.11) on J_T.

Lemma 1.

In any solution to the above problem, we have

$$M_t(x,0) \geq \omega > 0, \quad 0 \leq x \leq 2\pi \ . \tag{4.1}$$

Proof.

By assumption, a solution is a C^1 function on $\overline{J}_T$ for some $T > 0$. In particular, the system (2.11) must be satisfied as $t \downarrow 0$. By (2.12), the unit tangent to the initial curve, $\vec{\tau}(x,0)$, is well defined at every point. Also, by (2.13), the initial velocity, $\vec{U}(x,0)$, is never zero and is always to the left of $\vec{\tau}$. Hence, on $0 \leq x \leq 2\pi$,

$$\left. \frac{VR_x - UZ_x}{\left(R_x^2 + Z_x^2\right)^{1/2}} \right|_{t=0} = (\vec{\tau} \times \vec{U}) \cdot \vec{k} \geq \omega > 0, \tag{4.2}$$

where $\vec{k}$ is a unit vector in the x^3 direction. Using the last equation in (2.11), we obtain (4.1) from (4.2).

Lemma 2.

Let there exist a solution to the nonlinear problem. Then, there exists $T^* > 0$, such that

$$M(x,t) > 0 \text{ on } J_{T^*} \ . \tag{4.3}$$

Proof.

This follows from (2.14) and Lemma 1.

Lemma 3.

Let there exist a solution to the nonlinear problem and let T^* be as in Lemma 2. Then,

$$U^2 + V^2 = 1 \quad , \ (x,t) \in J_{T^*}. \tag{4.4}$$

Proof.

From the last three equations in (2.11), we have

$$\begin{cases} (MU)(MU)_t = -MUZ_x \\ (MV)(MV)_t = MVR_x \\ MM_t = MVR_x - MUZ_x \end{cases} \quad . \tag{4.5}$$

Hence,

$$\left(M^2\right)_t = \left(M^2V^2 + M^2U^2\right)_t \quad . \tag{4.6}$$

Using (2.14) we integrate (4.6) with respect to t to get

$$M^2(x,t) = M^2(x,t)\left[U^2(x,t) + V^2(x,t)\right] . \tag{4.7}$$

The result follows from Lemma 2.

Lemma 4.

The following two conditions on the initial data are necessary for existence of solutions,

$$U^2(x,0) + V^2(x,0) = 1 \tag{4.8}$$

and

$$U(x,0)R_x(x,0) + V(x,0)Z_x(x,0) = 0 . \tag{4.9}$$

Proof.

Equation (4.8) follows from (4.4) and continuity at $t = 0$. Equation (4.9) is a stronger requirement than (2.13) because it implies that the velocity must lie along the inward normal to the initial curve; (4.9) follows easily from the momentum equations at $t = 0$.

Lemma 5.

Knowledge of $U(x,t)$ and $V(x,t)$ in any rectangle, J_T, uniquely determines the other three dependent variables in J_T. We have

$$R(x,t) = R(x,0) + \int_0^t U(x,s)ds \ , \tag{4.10}$$

$$Z(x,t) = Z(x,0) + \int_0^t V(x,s)ds \ , \tag{4.11}$$

and

$$M(x,t) = g(x) \int_0^t \Big[U(x,s)U(x,0) + V(x,s)V(x,0)\Big] \, ds \ , \tag{4.12}$$

$$+ \int_0^t ds \int_0^s \Big[V(x,s)U_x(x,u) - U(x,s)V_x(x,u)\Big] \, du \ ,$$

where

$$g(x) = M_t(x,0) = \frac{R_x(x,0)}{V(x,0)} \geq \omega > 0. \tag{4.13}$$

Proof.

We need only establish the representation (4.12). Integrating the mass equation in (2.11), we get

$$M(x,t) = \int_0^t \Big[V(x,s)R_x(x,s) - U(x,s)Z_x(x,s)\Big] \, ds \ . \tag{4.14}$$

From (4.10) and (4.11), we have

$$R_x(x,s) = R_x(x,0) + \int_0^s U_x(x,u)du \tag{4.15}$$

and

$$Z_x(x,s) = Z_x(x,0) + \int_0^s V_x(x,u)du \ . \tag{4.16}$$

Next, from the momentum equations in (2.11), evaluated at $t = 0$,

$$M_t(x,0)U(x,0) = -Z_x(x,0) \tag{4.17}$$

and

$$M_t(x,0)V(x,0) = R_x(x,0) \ . \tag{4.18}$$

Define $g(x) = \dfrac{R_x(x,0)}{V(x,0)}$. Then, from (4.18) and (4.1), $g(x) \geq \omega > 0$, and from (4.17), $Z_x(x,0) = -g(x)U(x,0)$. By substituting (4.15) and (4.16) in (4.14), we obtain (4.12).

<u>Lemma 6</u>.

Let there exist a solution to the nonlinear problem and let T^* be as in Lemma 2. Then $U(x,t)$ and $V(x,t)$ are independent of t on J_T* .

<u>Proof</u>.

From Lemma 3, we have

$$U(x,t) = -\cos\,\theta(x,t) \tag{4.19}$$

and

$$V(x,t) = -\sin\theta(x,t) \quad , \tag{4.20}$$

where $\theta(x,t)$ is to be determined from its initial values $\theta(x,0)$. We now seek an evolution equation for $\theta(x,t)$. Using (4.19) and (4.20) in (4.12), we get

$$M(x,t) = g(x) \int_0^t \cos[\theta(x,s)-\theta(x,0)]ds$$

$$- \int_0^t ds \int_0^s \theta_x(x,u) \cos[\theta(x,s)-\theta(x,u)]du \quad . \tag{4.21}$$

From the last three equations in (2.11), we obtain

$$MU_t = -Z_x - M_t U = \left(U^2 -1\right) Z_x - UVR_x \tag{4.22}$$

and

$$MV_t = R_x - M_t V = \left(1-V^2\right) R_x + UVZ_x \quad . \tag{4.23}$$

Using (4.19) and (4.20) in (4.22), we obtain

$$M(x,t) \sin\theta(x,t) \frac{\partial\theta}{\partial t} = -\sin\theta(x,t) \Big[R_x(x,t) \cos\theta(x,t)$$

$$+ Z_x(x,t) \sin\theta(x,t)\Big] \quad . \tag{4.24}$$

Next, we substitute for R_x and Z_x, from (4.15) through (4.18) in (4.24), to obtain

$$M(x,t)\sin\theta(x,t)\frac{\partial\theta}{\partial t} = \sin\theta(x,t)\left\{-g(x)\sin[\theta(x,t)-\theta(x,0)] + \int_0^t \sin[\theta(x,t)-\theta(x,s)]\theta_x(x,s)ds\right\}. \tag{4.25}$$

Similarly, starting from (4.23), we obtain

$$M(x,t)\cos\theta(x,t)\frac{\partial\theta}{\partial t} = \cos\theta(x,t)\left\{-g(x)\sin[\theta(x,t)-\theta(x,0)] + \int_0^t \sin[\theta(x,t)-\theta(x,s)]\theta_x(x,s)ds\right\}. \tag{4.26}$$

Define $N(x,t)$ by

$$N(x,t) = -g(x)\sin[\theta(x,t)-\theta(x,0)] + \int_0^t \sin[\theta(x,t)-\theta(x,s)]\theta_x(x,s)ds. \tag{4.27}$$

Then, (4.25), (4.26) together yield the following evolution equation for $\theta(x,t)$.

$$M(x,t)\frac{\partial\theta}{\partial t} = N(x,t) \quad , \qquad (x,t) \in J_T{}^* \quad . \tag{4.28}$$

This nonlinear equation has a unique solution in $J_T{}^*$, namely, $\theta(x,t) = \theta(x,0)$. Using (4.21) and (4.27), we observe that

$$\frac{\partial N}{\partial t} = -\frac{\partial\theta}{\partial t}\frac{\partial M}{\partial t} \quad . \tag{4.29}$$

Hence, from (4.28) and (4.29),

$$M \frac{\partial N}{\partial t} + N \frac{\partial M}{\partial t} = 0 \quad \text{on } J_T^* \quad . \tag{4.30}$$

Consequently, from (4.30) and (2.14),

$$N(x,t)M(x,t) = N(x,0)M(x,0) = 0 \quad \text{on } J_T^* \quad . \tag{4.31}$$

Since $M(x,t) > 0$ on J_T^*, we conclude that $N(x,t) \equiv 0$ on J_T^*, and from (4.28), $\theta_t(x,t) \equiv 0$ on J_T^*. Hence,

$$U(x,t) = -\cos\theta(x) \tag{4.32}$$

and

$$V(x,t) = -\sin\theta(x) \tag{4.33}$$

as required.

Lemma 7.

Let there exist a solution to the nonlinear problem and let T^* be as in Lemma 2. Then, on J_T^*, this solution is given by

$$\begin{cases} U(x,t) = -\cos\theta(x) \\ V(x,t) = -\sin\theta(x) \\ R(x,t) = R(x,0) - t\cos\theta(x) \\ Z(x,t) = Z(x,0) - t\sin\theta(x) \\ M(x,t) = t\,g(x) - \dfrac{t^2}{2}\,\theta_x(x) \quad , \end{cases} \tag{4.34}$$

where $g(x) = \dfrac{R_x(x,0)}{-V(x,0)} \geq \omega > 0$, and $\theta(x) = \arccos[-U(x,0)]$.

Proof.

This follows immediately from (4.32), (4.33), and Lemma 5.

Lemma 8.

Let $\theta(x)$ be as in Lemma 7. Then, there exists an open interval $I \subset [0,2\pi]$, such that

$$\theta_x(x) > 0 \ , \quad x \in I \ . \tag{4.35}$$

Proof.

Let $\vec{i}$ and $\vec{j}$ be the unit vectors in the x^1 and x^2 directions, respectively, and $\vec{\tau}(x)$ be the unit tangent vector to the initial plasma-vacuum interface. Then, using (4.17), (4.18), (4.32), and (4.33),

$$\vec{\tau}(x) = \frac{\vec{i}\, R_x(x,0) + \vec{j}\, Z_x(x,0)}{\left[R_x^2(x,0) + Z_x^2(x,0)\right]^{1/2}} \tag{4.36}$$

$$= -\,\vec{i} \sin\theta(x) + \vec{j} \cos\theta(x) \ .$$

Let $\Psi(x)$ be the angle which $\vec{\tau}(x)$ makes with the x^1-axis. Then,

$$\vec{\tau}(x) \cdot \vec{i} = \cos\Psi(x) = -\sin\theta(x) \ . \tag{4.37}$$

Thus,

$$\Psi(x) = \theta(x) + \frac{\pi}{2} \ . \tag{4.38}$$

By hypothesis, the initial interface is a simple closed curve. Hence, the net increase in $\Psi(x)$ as x ranges from $x = 0$ to $x = 2\pi$ is exactly 2π. Therefore,

$$\int_0^{2\pi} \theta_x(x)dx = \theta(2\pi) - \theta(0) = 2\pi . \tag{4.39}$$

Since $\theta_x(x)$ is continuous on $[0,2\pi]$, the result follows.

Lemma 9.

Define $M(x,t)$ by

$$M(x,t) = tg(x) - \frac{t^2}{2}\theta_x(x) \tag{4.40}$$

for all $0 \leq x \leq 2\pi$ and all $t \geq 0$. Then, on any rectangle J_T , the set of points where $M(x,t) \neq 0$ is dense.

Proof.

For $t > 0$, $M(x,t) = 0$ if, and only if,

$$2g(x) = t\theta_x(x) . \tag{4.41}$$

Since $g(x) \geq \omega > 0$, we see from (4.41) that given any $x \in [0,2\pi]$, there is at most one positive value of t such that (4.41) is satisfied. Hence, in any rectangle J_T , the set of points where $M(x,t) = 0$ is either empty or lies on a curve. This proves the Lemma.

Theorem 1.

Let the initial data satisfy the necessary conditions, (4.8) and (4.9), in addition to (2.12), (2.13), and (2.14). Then, there exists a unique solution to the nonlinear initial value problem, (2.11). The solution exists for all $t \geq 0$ and is given by (4.34).

Proof.

It is readily verified that (4.34) is a solution of (2.11) for all $0 \leq x \leq 2\pi$ and all $t \geq 0$. According to Lemma 7, (4.34) is the only solution in any rectangle $J_T{}^*$ wherein $M(x,t) > 0$. An inspection of the proofs of Lemmas 3, 6, and 7 shows, however, that (4.34) is the only solution in any rectangle J_T in which $M^2(x,t) > 0$ on a dense subset. By Lemma 9, we conclude that (4.34) is the only solution to the problem on $t \geq 0$.

Theorem 2.

Let $R(x,t)$ and $Z(x,t)$ be as in (4.34) and let $\Gamma(t)$ be the curve in the (x^1,x^2) plane, defined parametrically by

$$x^1 = R(x,t) \tag{4.42}$$

and

$$x^2 = Z(x,t) \quad , \tag{4.43}$$

where t is fixed and $0 \leq x \leq 2\pi$.

Define T_c by

$$T_c = \inf_{x \in [0,2\pi]} \left\{ t(x) \,\middle|\, t(x) \geq 0,\ t(x) = \frac{g(x)}{\theta_x(x)} \right\}, \tag{4.44}$$

with $g(x)$ and $\theta(x)$ as in (4.34). Then $T_c > 0$, $\Gamma(t)$ develops a singularity as $t \uparrow T_c$, and $M(x,t) \leq 0$ for some $x \in [0,2\pi]$ if, and only if, $t \geq 2T_c$.

Proof.

By Lemma 8 and the fact that $g(x) \geq \omega > 0$, the set of points $t(x)$ such that $t(x) = \frac{g(x)}{\theta_x(x)}$ and $t(x) \geq 0$ is not empty. Since $\theta_x(x)$ is bounded on $[0,2\pi]$ and $g(x)$ is bounded away from zero, the above set of points $\{t(x)\}$ is also bounded away from zero. Hence $T_c > 0$. We now show that the trace of $\Gamma(t)$ develops a singular point as $t \uparrow T_c$. First, observe from (4.9) and (4.34) that for all $t \geq 0$,

$$U(x,t)R_x(x,t) + V(x,t)Z_x(x,t) = 0 \ . \tag{4.45}$$

From the mass equation in (2.11),

$$M_t(x,t) = V(x,t)R_x(x,t) - U(x,t)Z_x(x,t) \ . \tag{4.46}$$

Since $U^2(x,t) + V^2(x,t) = 1$, it follows from (4.45) and (4.46) that

$$R_x(x,t) = Z_x(x,t) = 0 \quad , \tag{4.47}$$

if, and only if, $M_t(x,t) = 0$. From (4.34), we have $M_t = g(x) - t\theta_x(x)$. Hence, the earliest time at which a singularity appears on the plasma-vacuum interface is given by T_c . Notice that from (4.41), the earliest time at which $M(x,t) = 0$ for some $x \in [0,2\pi]$ is $2T_c$.

Theorem 3.

(a) Let $g(x) - T_c\theta_x(x) \equiv 0$ on $[0,2\pi]$. Then, the trace of $\Gamma(t)$ is a circle whose radius shrinks to zero as $t \uparrow T_c$, and thereafter expands. (b) Let $g(x) - T_c\theta_x(x) = 0$ at some isolated point $x_0 \in [0,2\pi]$. Then, the trace of $\Gamma(T_c)$ has a continuously turning tangent at every point, but the radius of curvature tends to zero as $x \to x_0$. For sufficiently small $\varepsilon > 0$ and all t, such that $0 < (t-T_c) < \varepsilon$, the trace of $\Gamma(t)$ contains two cusps near x_0 . (c) Let $g(x) - T_c\theta_x(x) = 0$ on some closed interval [a,b] properly contained in $[0,2\pi]$. Then, the arc $a \leq x \leq b$ on the trace of $\Gamma(t)$, $t < T_c$ is an arc of circle which shrinks to a single point, γ, as $t \uparrow T_c$, and $\Gamma(t)$ develops a corner at γ . For t slightly greater than T_c , $\Gamma(t)$ contains two cusps, one near x = a and the other near x = b.

Proof of (a).

From (4.9), (4.13), and (4.34), we have

$$R_x(x,0) = -g(x) \sin\theta(x) = -T_c\theta_x(x) \sin\theta(x) \tag{4.48}$$

and

$$Z_x(x,0) = g(x) \cos \theta(x) = T_c \theta_x(x) \cos \theta(x) \; . \tag{4.49}$$

Hence,

$$R(x,0) = R(0,0) + T_c \int_0^x -\theta_u(u) \sin \theta(u) du \tag{4.50}$$

$$= R(0,0) + T_c[\cos \theta(x) - \cos \theta(0)] \; .$$

Similarly,

$$Z(x,0) = Z(0,0) + T_c[\sin \theta(x) - \sin \theta(0)] \; . \tag{4.51}$$

To complete the proof of (a) we substitute in (4.34) to obtain

$$R(x,t) - [R(0,0) - T_c \cos \theta(0)] = (T_c - t) \cos \theta(x) \tag{4.52}$$

$$Z(x,t) - [Z(0,0) - T_c \sin \theta(0)] = (T_c - t) \sin \theta(x) \; . \tag{4.53}$$

Proof of (b).

Let $\Psi(x,t)$ be the polar angle on $\Gamma(t)$, that is, the angle the unit tangent to $\Gamma(t)$, $\vec{\tau}(x,t)$, makes with the x^1-axis. From

$$\vec{\tau}(x,t) = \frac{\vec{i}\, R_x(x,t) + \vec{j}\, Z_x(x,t)}{\left[R_x^2(x,t) + Z_x^2(x,t)\right]^{1/2}} \; , \tag{4.54}$$

and (4.34), together with

$$R_x(x,0) = -g(x) \sin \theta(x) \; , \; Z_x(x,0) = g(x) \cos \theta(x), \tag{4.55}$$

we find

$$\cos \Psi(x,t) = \vec{\tau}(x,t) \cdot \vec{i} = \frac{-\sin \theta(x)\left[g(x) - t\theta_x(x)\right]}{|g(x) - t\theta_x(x)|} \tag{4.56}$$

Now if $t \leq T_c$ we have $g(x) - t\theta_x(x) \geq 0$ on $[0,2\pi]$, and hence from 4.56,

$$\Psi(x,t) = \theta(x) + \frac{\pi}{2} \ . \tag{4.57}$$

Also, the curvature κ is given by

$$\kappa(x,t) = \frac{\theta_x(x)}{|g(x) - t\theta_x(x)|} \ . \tag{4.58}$$

From (4.57) and (4.58) we see that if $g(x) - T_c\theta_x(x)$ vanishes at some isolated point x_0 , the trace of $\Gamma(T_c)$ has a continuously turning tangent at x_0, although the curvature there is infinite.

Next, let $\varepsilon > 0$ be sufficiently small and let $0 < (t-T_c) < \varepsilon$. Then $g(x_0) - t\theta_x(x_0) < 0$, and there are two points ξ and η, such that $\xi < x_0 < \eta$, and $g(x) - t\theta_x(x)$ changes sign at each of ξ and η. From (4.56), we see that these sign changes lead to a jump of π radians in the polar angle at ξ and η. This is the reason for the appearance of the two cusps near x_0 .

Proof of (c).

Consider $R(x,t)$ and $Z(x,t)$ for $a \leq x \leq b$ and $0 < t < T_c$. We have, using (4.55),

$$R(x,0) = R(0,0) - \int_0^a g(x) \sin\theta(x)dx + T_c[\cos\theta(x) - \cos\theta(a)], \tag{4.59}$$

$$Z(x,0) = Z(0,0) + \int_0^a g(x) \cos\theta(x)dx + T_c[\sin\theta(x) - \sin\theta(a)]. \tag{4.60}$$

Hence, using (4.34), we have,

$$R(x,t) = R(0,0) - T_c \cos\theta(a) - \int_0^a g(x) \cos\theta(x)dx + (T_c-t) \sin\theta(x), \tag{4.61}$$

$$Z(x,t) = Z(0,0) - T_c \sin\theta(a) + \int_0^a g(x)\cos\theta(x)dx + (T_c - t)\sin\theta(x), \qquad (4.62)$$
$$a \leq x \leq b \ .$$

Thus, as $t \uparrow T_c$, we have $R(a,t) \to R(b,t)$ and $Z(a,t) \to Z(b,t)$, and the arc $a \leq x \leq b$ on the trace of $\Gamma(t)$ is an arc of circle which shrinks to a single point, γ, as $t \uparrow T_c$. Moreover, from (4.57), we observe that at $t = T_c$, the polar angle at γ experiences a jump equal to $\theta(b) - \theta(a) \neq 0$, and hence $\Gamma(t)$ develops a corner at γ as $t \uparrow T_c$. Subsequently, this corner evolves into an arc of circle whose curvature is opposite to that when $t < T_c$. However, as in case (b) above, there will be two sign changes in $g(x) - t\theta_x(x)$, for t slightly greater than T_c, near $x = a$ and $x = b$, respectively. Hence, there will again be cusps at these points. This completes the proof of Theorem 3.

Theorem 4.

The unique solution (4.34) to the nonlinear initial value problem (2.11), although it exists for all $t \geq 0$, is not relevant to the underlying physical problem for $t \geq T_c$.

Proof.

Let $\mu(x,t)$ be the density of mass swept up by the piston as in (2.7), and let $S(x,t) = \left[R_x^2(x,t) + Z_x^2(x,t)\right]^{1/2}$ as in (2.6). Theorem 2 guarantees the existence of a point x_c in $[0,2\pi]$, such that $S(x_c,t)$ tends to zero as $t \uparrow T_c$, while $M(x,t_c) \geq \omega > 0$ on $[0,2\pi]$. Hence, since $\mu = M/S$, we have $\mu(x_c,t) \uparrow \infty$ as $t \uparrow T_c$, a physical impossibility.

Remarks.

The quantity T_c is an upper bound on the time interval during which the snow-plough model is physically realistic. In a real plasma, a magnetic piston will drive a shock which always travels faster than the piston. Thus, well before T_c, the shock front will develop singularities which lead to shock-shock interactions. These interactions generate a signal in the shocked material which travels back to the piston. The back signal modifies the magnetic piston behavior in a manner not included in the snow-plough model. Thus, even though the solution is completely regular up to $t = T_c$, the model breaks down somewhat earlier.

5. EXAMPLES

The simplest example of a solution to the nonlinear problem occurs when the initial interface is a circle. Let $A_0 > 0$ and let

$$R(x,0) = A_0 \cos x,\ Z(x,0) = A_0 \sin x, \tag{5.1}$$

$$U(x,0) = -\cos x,\ V(x,0) = -\sin x, \tag{5.2}$$

$$M(x,0) = 0, \tag{5.3}$$

where $0 \leq x \leq 2\pi$. In this case $g(x) = A_0$, $\theta(x) = x$, and

$$T_c = A_0 \ . \tag{5.4}$$

This corresponds to case (a) in Theorem 3. We have, from (4.34),

$$M(x,t) = A_0 t - \frac{t^2}{2}\ , \tag{5.5}$$

$$R(x,t) = (A_0-t)\cos x,\ Z(x,t) = (A_0-t)\sin x, \tag{5.6}$$

and the interface is a circle whose radius shrinks to zero as $t \uparrow A_0$.

Prior to that time, the above solution describes the following genuine physical situation. From (2.10),

$$\left(B_{vac}\right)^2 = \left(B_{p\ell}\right)^2 + 8\pi(p+1)\ , \tag{5.7}$$

where everything is measured in units consistent with $\Pi = \rho = 1$. The pressure p is constant and $B_{p\ell}$ is a uniform field parallel to the cylinder axis. As this field is trapped by the perfectly conducting magnetic piston, its total flux is conserved and we have

$$B_{p\ell}(t) = \left[A_0/A(t)\right]^2 B_0\ ;\quad B_0 \equiv B_{p\ell}(0), \tag{5.8a}$$

where

$$A(t) = \left[R^2(x,t) + Z^2(x,t)\right]^{1/2} = \left(A_0 - t\right) . \tag{5.8b}$$

Thus, (5.7) becomes

$$\left[B_{vac}(t)\right]^2 = 8\pi(p+1) + B_0^2 \left[1 - t/A_0\right]^{-4} . \tag{5.9}$$

Any programming of driving currents which produces such a vacuum field leads to a real snow-plough problem whose solution is given by (5.5) and (5.6). For example, if the vacuum is bounded by a metal shell of inner radius A_0, and if a purely axial current I(t) of the form

$$I(t) = \frac{A_0}{2} \left[\frac{8\pi(p+1)\left(1 - t/A_0\right)^4 + B_0^2}{\left(1 - t/A_0\right)^2}\right]^{1/2} \tag{5.10}$$

is driven along this shell, then such a current would drive a snow plough satisfying (5.5) and (5.6).

As another example, consider the following initial data

$$R(x,0) = 1 - \int_0^x \sin\left[u + \lambda \cos^2 u\right] du \tag{5.11}$$

$$Z(x,0) = \int_0^x \cos\left[u + \lambda \cos^2 u\right] du \tag{5.12}$$

$$U(x,0) = -\cos\left[x + \lambda \cos^2 x\right] \tag{5.13}$$

$$V(x,0) = -\sin\left[x + \lambda \cos^2 x\right] \tag{5.14}$$

$$M(x,0) = 0 , \tag{5.15}$$

where $0 \leq x \leq 2\pi$ and λ is fixed.

For values of λ such that

$$0 \leq \lambda \leq 1.5 \ , \tag{5.16}$$

the initial curve is a simple closed curve with a continuously turning tangent. Here,

$$g(x) = 1, \ \theta_x(x) = 1 - \lambda \sin 2x \tag{5.17}$$

and from (4.44),

$$T_c = \frac{1}{1+\lambda} \ . \tag{5.18}$$

With $\lambda = 1.3$, the smooth initial curve develops two singular points, at $x = 3\pi/4$ and $x = 7\pi/4$, as $t \uparrow T_c = 0.435$. This case corresponds to part (b) of Theorem 3. The evolution of the interface is shown in Fig. 2. Figures 2 and 3 were reproduced from computer plots where the scales on the vertical and horizontal axes are automatically adjusted, so that the resulting curve nearly fills the frame. In our case, this adjustment leads to a vertical magnification which greatly exceeds the horizontal magnification. The point $x = 3\pi/4$ in Figs. 2 and 3 is marked by a small square. At $t = T_c = 0.435$, we see that the curvature at $x = 3\pi/4$ appears rather modest in Fig. 2. We assure the reader that this is a consequence of the automatic scaling. The curvature at that point is infinite, although the tangent there turns continuously. The two cusps near $x = 3\pi/4$ are clearly visible in Fig. 2, for $t > T_c$, as well as the appearance of <u>multiple points</u>.

It is possible for multiple points to appear <u>before</u> $t = T_c$. Thus with $\lambda = 1.5$, the bottom part of the curve (near $x = 5\pi/4$) smoothly interlaces the top part (near $x = \pi/4$) at a value of $t < T_c = 0.4$, as shown in Fig. 3. This experiment confirms the fact that T_c is really an upper bound on the time during which the model is valid. In a real plasma, shock-shock interaction would have spoiled the model well before such an interlacing could occur.

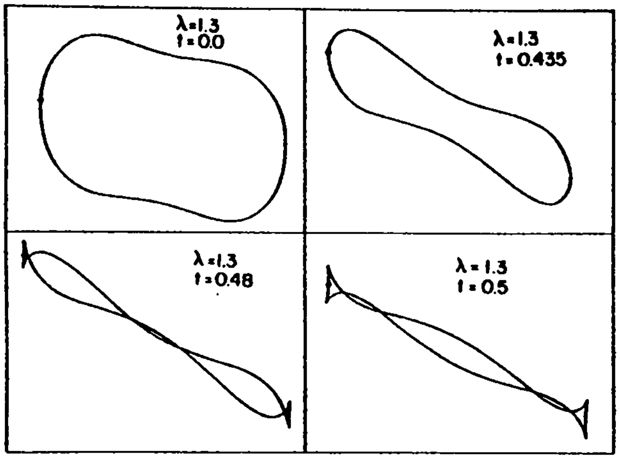

Fig. 2. Evolution of interface when $\lambda = 1.3$.

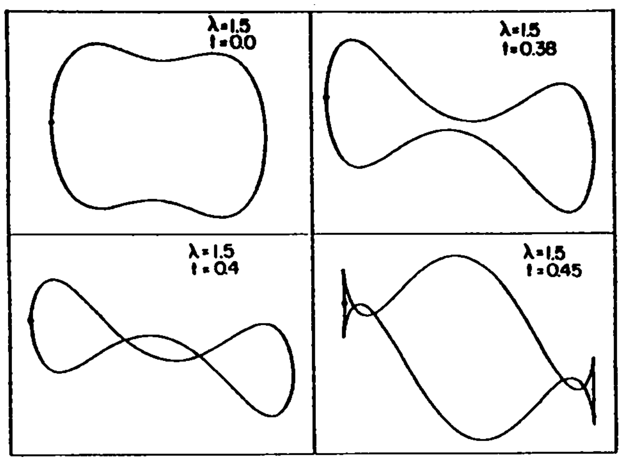

Fig. 3. Evolution of interface when $\lambda = 1.5$.

Although we do not show an example corresponding to part (c) of Theorem 3 in our figures, it is easy to see that such initial data exist. Thus, with $\lambda = 1.3$ as before, let

$$g(x) = 1, \; 0 \le x \le \frac{\pi}{2}, \; \pi \le x \le \frac{3\pi}{2}, \tag{5.19}$$

and

$$g(x) = 1 - \lambda \sin 2x, \; \frac{\pi}{2} \le x \le \pi, \; \frac{3\pi}{2} \le x \le 2\pi . \tag{5.20}$$

Define the initial interface by

$$R(x,0) = 1 - \int_0^x g(u) \sin\left[u + \lambda \cos^2 u\right] du , \tag{5.21}$$

$$Z(x,0) = \int_0^x g(u) \cos\left[u + \lambda \cos^2 u\right] du , \tag{5.22}$$

$0 \leq x \leq 2\pi$, so that,

$$\theta(x) = x + \lambda \cos^2 x. \tag{5.23}$$

Let $M(x,0) \equiv 0$ and let

$$U(x,0) = -\cos\theta(x), \quad V(x,0) = -\sin\theta(x). \tag{5.24}$$

It can be shown that the initial interface is a simple closed curve with a continuously turning tangent. We have $g(x) \geq 1$ on $[0,2\pi]$, $T_c = 1$, and $g(x) - T_c\theta_x(x) \equiv 0$ on $[\pi/2,\pi]$ and $[3\pi/2,2\pi]$. Accordingly, the interface develops corners at $t = 1$, with the polar angle making a jump of $(\lambda + \pi/2)$ rad at these points.

REFERENCES

1. M. Rosenbluth, "Infinite Conductivity Theory of the Pinch," Los Alamos Scientific Laboratory report LA-1850 (March 1955).

2. H. M. Nelson, K. H. Brown, and C. A. Hart, "Computer Model of A Fast Toroidal Plasma Compression, with Application to the Topolotron," Phys. Fluids 19, 1810-1819 (1976).

3. R. D. Richtmyer and K. W. Morton, Difference Methods for Initial Value Problems, 2nd Ed. (Wiley-Interscience, New York, 1967) p. 59.